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Yang–Mills fields invariant under subgroups of the Poincaré group

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Abstract. General SU(2) Yang-Mills configurations and solutions of the source-free Yang-Mills equations are investigated, which are invariant under some high subgroup of the Poincaré group. Simply transitive G_3, all multiply transitive G_3, multiply transitive G_4, which can be obtained by adding a fourth commuting Killing vector to a multiply transitive G_3, and also all groups G_r, $r \geq 6$ are considered. The results presented mainly concern the question of existence. Selected special invariant configurations are given explicitly.

1. Introduction

Solutions admitting a spacetime symmetry group play an important role in many physical theories. Detailed surveys on such solutions have been worked out for classical electrodynamics (Combe and Sorba 1975, Beckers *et al* 1978, 1979), Einstein's theory of gravity (Petrov 1969) and Dirac theory (Beckers *et al* 1981). Except for some special obvious symmetries, such as spherical symmetry, axial symmetry, translational invariance and a few others, a general survey on SU(2) Yang-Mills fields with spacetime symmetries has been missing up to now. This paper is intended as a step towards such a survey, dealing with Yang-Mills fields invariant under high subgroups of the Poincaré group.

The general problems connected with spacetime symmetries of gauge potentials have been solved during recent years, firstly for electrodynamics by Janner and Janssen (1971) and Giovannini (1977) and later for non-Abelian gauge fields by Bergmann and Flaherty (1978), Jackiw (1978) and Forgacs and Manton (1980). Forgacs and Manton, besides being the first to provide a general formulation of gauge potentials invariant under some spacetime symmetry, give an explicit rule for calculating these invariant potentials in the case of spherical symmetry. While their general formulation is applicable to any spacetime group, the calculational scheme—essentially resting on a change from the orbits of the group to the whole group—seems to be more restricted. Some premises such as semisimplicity of the spacetime group or independence of the components of the Killing vectors from the coordinates perpendicular to the orbits are not met for a general spacetime symmetry group. Therefore we decided to solve the equations on the orbits themselves by a method of successively fixing the gauge. This circumvents the restrictions mentioned and can be applied to any subgroup of the Poincaré group, yielding the most general invariant configuration.

The paper is organised as follows: after providing in § 2 some necessary background on spacetime symmetric gauge potentials we are ready to give in § 3 an outline of our method for calculating them. To illustrate this method an example is presented in § 4. Sections 5-8 give the results of applying this method to high subgroups of the Poincaré group, including some interesting whole classes of subgroups. In this paper we mainly concentrate on the question of the existence and general properties of the invariant potentials obtained, while some special results are given elsewhere (Basler and Hädicke 1984a, b).

We should point out that our investigations resting on the formulation of Forgacs and Manton (1980) are purely local. There is an alternative global approach to spacetime symmetric gauge fields, developed essentially by Harnad and Vinet (1978) and Harnad *et al* (1979, 1980). Although this global formulation is of considerable mathematical beauty the local description with the Lie derivative seems to be better adapted for the explicit calculation of invariant potentials. After all, Harnard and Vinet (1978), Harnad *et al* (1979) and Beckers and Hussin (1984) have cleared up the relations between both approaches.

2. Spacetime symmetric gauge fields

According to Forgacs and Manton (1980) a gauge field admits a spacetime symmetry, if the spacetime transformation of the potential can be compensated by a gauge transformation. In the SU(2) Yang-Mills theory this leads to the symmetry equations

$$\mathscr{L}_{\xi_n} A_\mu = D_\mu W_n. \tag{2.1}$$

We adopt matrix notation

$$A_\mu = A_\mu^A T^A \qquad T^A = \tfrac{1}{2}\sigma^A$$

where σ^A are the Pauli matrices. The field strengths are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]. \tag{2.2}$$

$\mathscr{L}_{\xi_n}$ means the Lie derivative along one of the Killing vectors ξ_n $(n = 1, \ldots, r)$ of the spacetime symmetry group G_r under consideration. The so-called symmetry potentials W_n characterise the compensating gauge transformation. The ξ_n fulfil the commutation relations

$$[\xi_m, \xi_n] = c^k_{mn}\xi_k. \tag{2.3}$$

If we have only one Killing vector the symmetry equation (2.1) can be easily solved by choosing a gauge where $W = 0$. Possible higher symmetries are provided by the subgroups of the Poincaré group, because the Yang-Mills equations are Poincaré invariant (indeed they are conformal invariant even). Complete lists of all subgroups of the Poincaré group have been developed by Laßner (1973), Bacry *et al* (1974) and Patera *et al* (1975), where the latter paper has been taken as the reference here.

By further Lie differentiating (2.1) along a second Killing vector ξ_m and taking the antisymmetric part one arrives at a consistency equation for the symmetry potentials:

$$\mathscr{L}_{\xi_m} W_n - \mathscr{L}_{\xi_n} W_m - \mathrm{i}g[W_m, W_n] - c^k_{mn} W_k - K_{mn} = 0 \tag{2.4}$$

$$D_\mu K_{mn} = 0. \tag{2.5}$$

Up to now, in most investigations the covariant constants K_{mn} have been omitted. A systematic treatment for the U(1) case is given by Beckers and Hussin (1984). For the SU(2) case by diagonalising one of the K_{mn} and further utilising (2.5) one finds that there are two possible cases:

(i) if one of the K_{mn} is different from zero, only embedded Abelian fields exist;

(ii) proper non-Abelian fields only exist if all K_{mn} vanish.

Thus, whereas the case $K_{mn}=0$ leads to the most general non-Abelian field, the most general Abelian (embedded U(1)) field can be found by taking all K_{mn} as constants different from zero. In this connection we should remark that for higher groups SU(N), $N>2$ even $K_{mn}\neq 0$ may lead to non-Abelian fields.

3. The calculation of gauge invariant potentials

To calculate the invariant potentials for a given subgroup of the Poincaré group we have to find the general solution of (2.4) and put the W_n obtained into equations (2.1). These can be solved afterwards to give the most general invariant potentials for this group.

Our calculational method consists of three steps. We start with a suitable choice of coordinates which are fitted to the symmetry under consideration. These coordinates

$$x^{\mu'}=(e, a, b, c) \tag{3.1}$$

are subdivided into two classes

x^{α}	tangential to the orbits
x^{i}	normal to the orbits

of the spacetime symmetry group. Because these orbits are the integral surfaces of the Killing vectors, a choice is possible where the Killing vectors have only derivatives along the orbits of the group

$$\mathscr{L}_{\xi_m} W_n=\xi^{\mu'}_m\,\partial_{\mu'} W_n=\xi^{\alpha}_m\,\partial_\alpha W_n. \tag{3.2}$$

In particular one Killing vector can be given the form

$$\xi_1=\partial_a \tag{3.3}$$

by choosing its integral curves as coordinate lines a. The possible forms of the further Killing vectors depend on the structure of the spacetime group. Using the normal form classification of Killing vectors given by Petrov (1969) and the table of Poincaré subgroups from Patera *et al* (1976) it can be shown that the following normal forms are possible for Poincaré subgroups G_r, $r\geqslant 3$, to which our investigations are confined:

simply transitive G_3

$$\begin{aligned}
&\xi_1=\partial_a\\
&\xi_2=A_2(b, c, e)\,\partial_a+\partial_b\\
&\xi_3=A_3(a\ldots e)\,\partial_a+B_3(a\ldots e)\,\partial_b+C_3(c)\,\partial_c\\
&C_3\neq 0
\end{aligned} \tag{3.4}$$

multiply transitive G_3

$$\begin{aligned}
&\xi_1 = \partial_a \\
&\xi_2 = A_2(a\ldots e)\,\partial_a + B_2(a\ldots e)\,\partial_b \\
&\xi_3 = A_3(a\ldots e)\,\partial_a + B_3(a\ldots e)\,\partial_b \\
&B_2 \neq 0 \qquad B_3 \neq 0 \qquad A_2^2 + A_3^2 \neq 0
\end{aligned} \tag{3.5}$$

simply transitive G_4

$$\begin{aligned}
&\xi_1 = \partial_a \\
&\xi_2 = A_2(b, c, e)\,\partial_a + \partial_b \\
&\xi_3 = A_3(a\ldots e)\,\partial_a + B_3(a\ldots e)\,\partial_b + C_3(c)\,\partial_c \\
&\xi_4 = A_4(a\ldots e)\,\partial_a + B_4(a\ldots e)\,\partial_b + C_4(a\ldots e)\,\partial_c + E_4(e)\,\partial_e \\
&E_4 \neq 0
\end{aligned} \tag{3.6}$$

multiply transitive G_4

$$\begin{aligned}
&\xi_1 = \partial_a \\
&\xi_2 = A_2(b, c, e)\,\partial_a + \partial_b \\
&\xi_3 = A_3(a\ldots e)\,\partial_a + B_3(a\ldots e)\,\partial_b + C_3(a\ldots e)\,\partial_c \\
&\xi_4 = A_4(a\ldots e)\,\partial_a + B_4(a\ldots e)\,\partial_b + C_4(a\ldots e)\,\partial_c \\
&C_3^2 + C_4^2 \neq 0.
\end{aligned} \tag{3.7}$$

For higher subgroups, which act on the whole Minkowski space (up to two groups G_5 not considered here) we start from a G_4 and add the fifth Killing vector. Because every Poincaré subgroup G_5 has a subgroup G_4 and every G_6 has a subgroup G_5 or G_4 this does not present any difficulties.

After having transformed the Killing vectors we are ready to solve in a second step the consistency equations, which are now formulated on the orbits

$$\xi_m^\alpha\,\partial_\alpha W_n - \xi_n^\alpha\,\partial_\alpha W_m - \mathrm{i}g[W_m, W_n] - c_{mn}^k W_k = 0 \tag{3.8}$$

for $K_{mn} = 0$

$$\xi_m^\alpha\,\partial_\alpha W_n - \xi_n^\alpha\,\partial_\alpha W_m - c_{mn}^k W_k - K_{mn} = 0 \qquad W_n = W_n^3 T^3 \tag{3.9}$$

for $K_{mn} \neq 0$. Thereby we exploit the gauge freedom of the symmetry potentials

$$W_n^\omega = \omega W_n \omega^{-1} - (\mathrm{i}/g)(\xi_n^\alpha\,\partial_\alpha \omega)\omega^{-1}. \tag{3.10}$$

From (3.3) we immediately see that W_1 can be gauged away by a transformation ω_1

$$W_1^{\omega_1} = 0 \tag{3.11}$$

because

$$\omega_1 W_1 = (\mathrm{i}/g)\,\partial_a \omega_1$$

can be solved by

$$\omega_1 = \mathscr{P}\exp\left(\frac{g}{\mathrm{i}}\int_0^a W_1(\alpha, b, c, e)\,\mathrm{d}\alpha\right) \tag{3.12}$$

in any case. With (3.11) the first $(r-1)$ consistency equations (3.8) (equations (3.9) for $K_{mn} \neq 0$ can be dealt with in a similar way) are

$$\partial_a W_n - c_{1n}^k W_k = 0. \tag{3.13}$$

This is a linear homogeneous system of ordinary differential equations with constant coefficients, which can be easily solved to determine the a dependence of the symmetry potentials W_n, $n = 1 \ldots r$. Now we consider the gauge transformations of W_2

$$W_2^{\omega_2} = \omega_2 W_2 \omega_2^{-1} - (\mathrm{i}/g)(\xi_2^\alpha \partial_\alpha \omega_2)\omega_2^{-1}. \tag{3.14}$$

To preserve (3.11) we are restricted to a residual gauge freedom

$$\omega_2 = \omega_2(b, c, e). \tag{3.15}$$

The further treatment obviously depends on the structure obtained for $W_2(a, b, c, e)$ and on $\xi_2^b = B_2(a \ldots e)$. In what follows we make use of the fact that all but two of the subgroups G_r, $r \geq 3$ of the Poincaré group have an Abelian subgroup G_2 for their part. If this G_2 is constituted by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, from (3.13) we conclude

$$W_2 = W_2(b, c, e). \tag{3.16}$$

On the other hand, if $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ commute, $\boldsymbol{\xi}_2$ can be given the form

$$\boldsymbol{\xi}_2 = A_2(b, c, e)\, \partial_a + B_2(b, c, e)\, \partial_b \tag{3.17}$$

(cf (3.4), respectively (3.5); the only two exceptions evidently are two multiply transitive G_3). Under these conditions the equation

$$\omega_2(b, c, e)\, W_2(b, c, e) = (\mathrm{i}/g) B_2(b, c, e)\, \partial_b \omega_2(b, c, e)$$

can be solved by

$$\omega_2 = \mathscr{P} \exp\left(\frac{g}{\mathrm{i}} \int_0^b \frac{W_2(\beta, c, e)}{B_2(\beta, c, e)}\, \mathrm{d}\beta\right) \tag{3.18}$$

to give

$$W_2^{\omega_2} = 0. \tag{3.19}$$

(Note that $B_2 \neq 0$ in all cases). By this choice of gauge we have simplified the equations for W_3 considerably:

$$\begin{aligned} \partial_a W_3 - c_{13}^3 W_3 &= 0 \\ \partial_b W_3 - c_{23}^3 W_3 &= 0. \end{aligned} \tag{3.20}$$

From (3.20) we find the general form for W_3

$$W_3 = \Phi_3(c, e) \exp(c_{13}^3 a + c_{23}^3 b). \tag{3.21}$$

Thus, for a G_3 the consistency equations (3.8) are solved. Without going into details, we mention that the two multiply transitive G_3 without an Abelian subgroup G_2 can be dealt with in a similar way. The W_n for higher subgroups can be constructed starting from one of the G_3 and adding the further Killing vectors. From the additional consistency equations the W_n, $n > 3$ can be calculated easily, because the structure of W_1, W_2 and W_3, which enter these equations, is already known. Additional simplification can be obtained by exploiting the still remaining gauge freedom, as was done for W_2 and W_3.

This scheme of successive gauging and solving can be also applied to the case $K_{mn} \neq 0$, where all W_n and K_{mn} in (3.9) commute. The difference is that, because of the $K_{mn} \neq 0$ in (3.9), even for $c_{12}^k = 0$ we cannot conclude $\partial_a W_2 = 0$ from $W_1 = 0$. Thus, the residual gauge freedom cannot be used to gauge away W_2 completely (thereby preserving $W_1 = 0$), but only to make a free function of b, c and e to zero. Consequently, the determination of the W_n for $K_{mn} \neq 0$ needs a bit more calculation, but in principle it can be done the same way.

After determining the symmetry potentials W_n we are ready to change over to the symmetry equations (2.1) in a third step. With coordinates (3.1), the equations (2.1) are also formulated on the orbits

$$\mathscr{L}_{\xi_n} A_{\mu'} = \xi_n^\beta \, \partial_\beta A_{\mu'} + (\partial_{\mu'} \xi_n^\beta) A_\beta = D_{\mu'} W_n. \qquad (3.22)$$

Because of

$$W_1 = 0$$

and

$$\boldsymbol{\xi}_1 = \partial_a$$

we conclude that in every case the gauge potentials $A_{\mu'}$ are independent of at least one coordinate

$$A_{\mu'} = A_{\mu'}(b, c, e). \qquad (3.23)$$

The remaining equations are of two types,
(i) differential equations
(ii) algebraic equations
for certain components A_{μ}. In solving them we restrict ourselves to solutions with C^∞ functions. Also for (3.22) the cases $K_{mn} = 0$ and $K_{mn} \neq 0$ must be treated separately with $A_{\mu'} = A_{\mu'}^3 T^3$ in the last case. In this way we get all possible invariant configurations for a given spacetime symmetry group. An interesting application of the configurations obtained consists in putting them into the source-free Yang–Mills equations to find out under which additional restrictions these can be fulfilled. In several cases it turns out that in spite of the existence of general configurations solutions of the source-free equations are excluded. On the other hand, some embedded Abelian solutions (embedded Maxwell fields) have k additional Killing vectors, which enlarge the symmetry group of the solution from the presumed G_r to a G_{r+k}.

4. An example

We consider the following group

$$\{P_1, P_0 - P_3, L_2 + K_1\} \qquad (4.1)$$

as an example for our calculational scheme, restricting ourselves to $K_{mn} = 0$ again. In characterising the subgroups of the Poincaré group the following notation is used:

L_i rotation around the i axis

K_i Lorentz boost into the i direction

P_μ translation into the μ direction

(in particular P_0 is a timelike translation). Thus, the group (4.1) consists of a spacelike translation, a null-like translation and a so-called null rotation, consisting of a combination of a rotation and a boost. The orbits of this group are two-dimensional null surfaces, thus the group acts multiply transitive. To obtain the corresponding form (3.5) we can change from the cartesian coordinate where

$$\begin{aligned} \xi_1 &= P_1 = \partial_x \\ \xi_2 &= P_0 - P_3 = -(\partial_z + \partial_t) \\ \xi_3 &= L_2 + K_1 = (t-z)\,\partial_x + x(\partial_z + \partial_t) \end{aligned} \tag{4.2}$$

to

$$x^\alpha : a = x \qquad b = -\tfrac{1}{2}(z+t)$$

$$x^i : c = y \qquad e = 2^{-1/2}(z-t).$$

In these coordinates the Killing vectors only have derivatives along a and b

$$\begin{aligned} P_1 &= \partial_a \\ P_0 - P_3 &= \partial_b \\ L_2 + K_1 &= -\sqrt{2}e\,\partial_a - a\,\partial_b. \end{aligned} \tag{4.3}$$

Now we insert (4.3) into the consistency equations (3.8). We notice that the only non-vanishing structure constant is $c_{13}^2 = -1$. After gauging

$$W_1 = 0$$

for $n = 2$ equation (3.13) gives

$$W_2 = W_2(b, c, e).$$

Because of

$$\xi_2 = \partial_b$$

which means that $A_2 = 0$ and $B_2 = 1$ in (3.17) the conditions are fulfilled to gauge further

$$W_2 = 0.$$

Next, because of $c_{13}^3 = c_{23}^3 = 0$ (3.21) reduces to

$$W_3 = \Phi_3(c, e).$$

Because of

$$\xi_3 \omega_3(c, e) = 0$$

the residual gauge freedom for W_3

$$W_3^{\omega_3} = \omega_3(c, e)\,W_3(c, e)\omega_3(c, e)^{-1} - (\mathrm{i}/g)(\xi_3\omega_3)\omega_3^{-1}$$

cannot be used to gauge $W_3 = 0$ too. Thus, we remain with

$$W_1 = W_2 = 0 \qquad W_3 = \Phi_3(c, e). \tag{4.4}$$

With this, the symmetry equations (3.22) read

$$\partial_a A_{\mu'} = 0 \tag{4.5}$$

$$\partial_b A_{\mu'} = 0 \tag{4.6}$$

$$-\sqrt{2}(\partial_{\mu'} e)A_a - (\partial_{\mu'} a)A_b = \partial_{\mu'}\phi_3 - \mathrm{i}g[A_{\mu'}, \Phi_3]. \tag{4.7}$$

From (4.5) and (4.6) we have

$$A_{\mu'} = A_{\mu'}(c, e).$$

Now we have to distinguish the cases $\Phi_3 = 0$ and $\Phi_3 \neq 0$. For $\Phi_3 = 0$, we see that

$$A_a = A_b = 0.$$

Exploiting the residual gauge freedom $\omega = \omega(c, e)$, we still can choose a gauge where

$$A_e = 0$$

and the gauge potentials are

$$A_e = A_a = A_b = 0 \qquad A_c = g(c, e). \tag{4.8}$$

There is only one non-vanishing component of the field strength tensor

$$F_{ec} = \partial_e g(c, e). \tag{4.9}$$

It should be noted that this configuration is still a non-Abelian one, because the commutator

$$[A_c, F_{ec}] = [g, \partial_e g]$$

which enters the field equations, is not obliged to vanish in general.

On the other hand, for $\Phi_3 \neq 0$ we use the residual gauge freedom $\omega(c, e)$ to diagonalise

$$W_3 = \Phi_3^3(c, e) T^3. \tag{4.10}$$

Then, from an analysis of (4.5)-(4.7) we learn

$$A_e = g_1(c, e) T^3 \qquad A_a = A_b = 0 \qquad A_c = g_2(c, e) T^3. \tag{4.11}$$

This is an embedded Abelian configuration. However the most general embedded Abelian configuration will be obtained for $K_{mn} \neq 0$, thus (4.11) will only be a special case of the potentials obtained then.

The formalism described above was applied to high subgroups G_r, $r \geqslant 3$ of the Poincaré group, including particularly:

(i) groups with an obvious interpretation and/or physical applicability;

(ii) some interesting whole classes of subgroups.

In what follows we will quote some of the general results, concentrating mainly on the question of existence.

5. Simply transitive groups G_3

The class of simply transitive subgroups G_3 of the Poincaré group is rather extensive, thus we selected only a few of them. Table 1 shows the general results.

The notation for the groups is as in § 3. The orbits are denoted as follows:

$$\left.\begin{matrix} N_s \\ S_s \\ T_s \end{matrix}\right\} \text{denotes a} \left\{\begin{matrix} \text{null-like} \\ \text{space-like} \\ \text{time-like} \end{matrix}\right\} s\text{-dimensional hypersurface}$$

'Conf.' stands for general SU(2) configuration, whereas 'Solution' refers to solutions of the source-free Yang-Mills equations. The symbol ∃ means that at least one

Table 1. Selected Yang-Mills configurations and solutions of the source-free Yang-Mills equations invariant under a simply transitive G_3.

		Abelian		non-Abelian	
Group	Orbits	Conf.	Solutions	Conf.	Solutions
P_1, P_2, P_0-P_3	N_3	∃	∃	∃	—
P_1, P_2, P_3	S_3	∃	∃	∃	∃
P_1, P_2, P_0	T_3	∃	∃	∃	∃
$P_0-P_3, L_2+K_1, L_1-K_2$	N_3	∃	∃	∃	—
P_0-P_3, P_2, L_2+K_1	N_3	∃	∃	∃	—

configuration or solution with this symmetry exists; — means that the existence of such a configuration/solution could be excluded.

For the first three groups the Abelian configurations are well known; they are embeddings of the corresponding electromagnetic fields. For instance for P_1, P_2, P_0-P_3 we find (in the case of $K_{mn} \neq 0$)

$$A_e = 0 \qquad A_a = (K_{12}b + K_{13}c + g_1(e))T^3$$
$$A_b = (K_{23}c + g_2(e))T^3 \qquad A_c = g_3(e)T^3 \tag{5.1}$$

where

$$a = x \qquad b = y \qquad c = -\tfrac{1}{2}(z+t) \qquad e = \tfrac{1}{2}(z-t).$$

Inserting this configuration into the Yang-Mills equations the non-Abelian terms clearly vanish and we obtain the additional restriction for source-free solutions

$$g_3 = K_1 e + K_2.$$

The corresponding field strengths can be calculated and are a superposition of a plane-wave part and an additional constant contribution.

The more interesting results concern the non-Abelian configurations. For the same group these are

$$A_e = 0 \qquad A_a = g_1(e) \qquad A_b = g_2(e) \qquad A_c = g_3(e) \tag{5.2}$$

where now the g_n are general SU(2) matrices

$$g_n = g_n^A T^A.$$

The general structure (5.2) is common to the first three groups in table 1. We obtain potentials, which depend on either one null-like or one spacelike or one timelike coordinate only. The corresponding solutions of the source-free Yang-Mills equations for such potentials have already been investigated by Raczka (1982). The most remarkable result is the non-existence of non-Abelian source-free solutions if this coordinate is a null-like one. On the contrary, explicit solutions were constructed by Raczka for the case of a spacelike or timelike coordinate.

The group $\{P_1, P_2, P_0-P_3\}$ and also the last two in table 1 are subgroups of the symmetry group of the plane waves in electrodynamics

$$A_e = A_c = 0 \qquad A_a = g_1(e) \qquad A_b = g_2(e)$$
$$a = x \qquad b = y \qquad c = -\tfrac{1}{2}(z+t) \qquad e = \tfrac{1}{2}(z-t) \tag{5.3}$$

which is the group $\{P_1, P_2, P_0 - P_3, L_2 + K_1, L_1 - K_2\}$. Already from this it follows that non-Abelian solutions of the source-free Yang–Mills equations with the same symmetry do not exist. Moreover, also non-Abelian 'non-plane wave' solutions (in the sense of being invariant against $L_2 + K_1$, $L_1 - K_2$, $P_0 - P_3$ only) are excluded herewith.

Another application of classical Yang–Mills fields concerns an analogy with the stochastic movement in classical mechanics ('Yang–Mills mechanics', see Baseyan *et al* 1979, Matinyan *et al* 1981). The non-Abelian configurations used there are found to be invariant under $\{P_1, P_2, P_3\}$.

6. All multiply transitive groups G_3

Because this class of subgroups covers a lot of interesting cases, it was investigated completely.

Table 2. Yang–Mills configurations and solutions of the source-free Yang–Mills equations invariant under a multiply transitive G_3.

		Abelian		non-Abelian	
Group	Orbits	Conf.	Solutions	Conf.	Solutions
$P_1, P_0 - P_3, L_2 + K_1$	N_2	∃	∃ $(+P_2, L_1 - K_2)$	∃ Type I	—
P_1, P_2, L_3	S_2	∃	∃ $(+P_3, P_0, K_3)$	∃ Type I ∃ Type II	— ∃
P_3, P_0, K_3	T_2	∃	∃ $(+P_1, P_2, L_3)$	∃ Type I	—
$L_2 + K_1, L_1 - K_2, L_3$	S_2	∃	∃ $(+P_0 - P_3)$	∃ Type I ∃ Type II	— ?
L_1, L_2, L_3	S_2	∃	∃ $(+P_0)$	∃ Type I ∃ Type II	— ∃
K_1, K_2, L_3	$S_2 - N_2 - T_2$	∃	∃ $(+P_3)$	∃ Type I	—

Firstly, we record that all embedded Abelian solutions possess further Killing vectors, stated in other words, Abelian solutions with a maximal multiply transitive G_3 do not exist. In particular—as is well known for electrodynamics—all spherically symmetric embedded Abelian solutions of the source-free field equations have to be static. Remarkably, this is not true for non-Abelian solutions. There exist explicit examples of time-dependent spherically symmetric non-Abelian solutions of the source-free Yang–Mills equations (de Alfaro *et al* 1976, Actor 1972).

For the non-Abelian configurations we find in general two different, non-gauge-equivalent types of fields. This fact is already well known for spherical symmetry (Malec 1982, Jackiw 1980, Miyachi *et al* 1982), but obviously is a more general phenomenon. Hence, type-I configurations are characterised by the fact that all symmetry potentials can be gauged to zero simultaneously, which is impossible for type-II configurations. As can be read off in table 2, such type-II configurations exist only for those groups, which act on a spacelike 2-surface S_2. We further discover, that source-free solutions are obtainable from type-II configurations only.

We add some remarks concerning special subgroups. $\{P_1, P_2, L_3\}$ is the symmetry group of a plane. The embedded Abelian fields—as one would guess from electrodynamics—consists of a magnetic and an electric field perpendicular to the plane. After exploiting the field equations the fields have to be constant. There are two types of non-Abelian configurations:

type I

$$A_t = g_1(z, t) \qquad A_x = A_y = 0 \qquad A_z = g_2(z, t) \tag{6.1}$$

type II

$$\begin{aligned} &A_t = Q(z, t)T^3 \qquad A_x = R(z, t)T^1 - S(z, t)T^2 \\ &A_y = S(z, t)T^1 + R(z, t)T^2 \qquad A_z = P(z, t)T^3. \end{aligned} \tag{6.2}$$

One of the functions Q or P can still be gauged to zero by means of the remaining gauge freedom. As we see in table 2, solutions of the source-free equations can be obtained from (6.2) only. Further details on these solutions can be found in another paper (Basler and Hädicke 1984b).

For the group $\{L_2 + K_1, L_1 - K_2, L_3\}$ there is a non-Abelian type-II configuration, so we could guess that non-Abelian solutions of the source-free field equations exist. Unfortunately, since the resulting field equations for this symmetry are rather complicated, neither could we construct an explicit solution, or exclude its existence.

$\{L_1, L_2, L_3\}$ is the rotational group which has been studied extensively in the past (Witten 1977, Gu 1981, and others). For comparison with the group $\{K_1, K_2, L_3\}$, which looks similar to the rotational group, we quote the results:

coordinates

$$\begin{aligned} &a = \varphi = \tan^{-1} x/y \qquad b = \vartheta = \tan^{-1} z/(x^2+y^2)^{-1/2} \\ &c = r = (x^2+y^2+z^2)^{1/2} \qquad e = t = t. \end{aligned}$$

Abelian configuration

$$\begin{aligned} &A_t = g(r, t)T^3 \qquad A_x = -C\frac{\cos\varphi \sin\vartheta}{r\cos\vartheta}T^3 \\ &A_y = C\frac{\sin\varphi\sin\vartheta}{r\cos\vartheta}T^3 \qquad A_z = 0. \end{aligned} \tag{6.3}$$

Non-Abelian type-I configuration

$$A_t = g_1(r, t) \qquad A_\varphi = A_\vartheta = 0 \qquad A_r = g_2(r, t). \tag{6.4}$$

Non-Abelian type-II configuration

$$\begin{aligned} &A_t = 0 \qquad A_\varphi = -(1/g)\sin\vartheta T^3 + \cos\vartheta(RT^1 - ST^2) \\ &A_\vartheta = -RT^2 - ST^1 \qquad A_r = PT^3 \\ &P = P(r, t) \qquad R = R(r, t) \qquad S = S(r, t). \end{aligned} \tag{6.5}$$

To recover the Dirac string for $\vartheta = \pi/2$ explicitly, we have converted to cartesian components of the potentials for the Abelian configuration.

We want to compare this with the configurations for $\{K_1, K_2, L_3\}$. Here the coordinates chosen are

$$a = \psi = \tan^{-1}\frac{x}{y} \qquad b = \psi = \tanh^{-1}\frac{t}{x^2+y^2}$$
$$c = s = (x^2+y^2-t^2)^{1/2} \qquad e = z = z.$$

We get the following Abelian configuration

$$A_t = g(s,t)T^3 \qquad A_x = -C\frac{\cos\psi\sinh\chi}{s\cosh\chi}T^3$$
$$A_y = C\frac{\sin\psi\sinh\chi}{s\cosh\chi}T^3 \qquad A_z = 0. \tag{6.6}$$

At first glance, this looks similar to (6.3), but in (6.6) the string corresponding to $\cos\vartheta = 0$ is missing; instead we recover a singularity on the hypersurface $x^2+y^2-t^2=0$. The non-Abelian type-I configuration reads here

$$A_z = g_1(s,t) \qquad A_\psi = A_\chi = 0 \qquad A_z = g_2(s,t). \tag{6.7}$$

As in the case of the rotational group this leads to non-Abelian solutions of the source-free equations only for external sources. However it is well known that non-Abelian source-free solutions can be gained for the rotational group from the type-II configuration (Wu and Yang 1969 and others) which are even static additionally. Thus, it is somewhat surprising, that type-II configurations for the group $\{K_1, K_2, L_3\}$ do not exist at all.

7. Multiply transitive groups G_4

The multiply transitive G_4 selected here come out by adding a fourth commuting Killing vector to one of the multiply transitive G_3. The general results are given in table 3.

Table 3. Selected Yang-Mills configurations and solutions of the source-free Yang-Mills equations invariant under a multiply transitive G_4.

		Abelian		non-Abelian	
Group	Orbits	Conf.	Solutions	Conf.	Solutions
P_1, P_2, P_0-P_3, L_2+K_1	N_3	∃	∃	Type I	—
P_1, P_2, P_3, L_3	S_3	∃	∃	Type I	—
				Type II	∃
P_1, P_2, P_0, L_3	T_3	∃	∃	Type I	—
				Type II	∃
P_0-P_3, L_2+K_1, L_1-K_2, L_3	N_3	∃	∃	Type I	—
				Type II	—
P_0, L_1, L_2, L_3	T_3	∃	∃	Type I	—
				Type II	∃
P_3, K_1, K_2, L_3	$S_3-N_3-T_3$	∃	∃	Type I	—

The case of spherical symmetry and statics has been studied extensively. Besides the embedded Abelian (Coulomb) solution one can get solutions of the source-free field equations only from the type-II configurations. This leads to the Wu–Yang equations (Wu and Yang 1968, for a recent investigation see Hädicke and Pohle 1983). It is not surprising that the similar group $\{K_1, K_2, L_3, P_3\}$ which does not lead to a type-II configuration does not allow for a non-Abelian solution at all.

The two groups $\{P_1, P_2, P_3, L_3\}$ and $\{P_1, P_2, P_0, L_3\}$ describe a plane, which has an additional symmetry: in the first case the solution has to be independent of the distance to the plane, in the second case it has to be static additionally. In this last case, the type-II configuration leads to interesting non-Abelian solutions of the source-free equations, for instance to the one

$$A_t = 0 \qquad A_x = \frac{\sqrt{2}}{gz} T^1 \qquad A_y = \frac{\sqrt{2}}{gz} T^2 \qquad A_z = 0. \tag{7.1}$$

Further solutions, implying elliptic functions, are given elsewhere (Basler and Hädicke 1984b).

Remarkably, the group $\{P_0, P_3, L_2+K_1, L_1-K_2, L_3\}$, which allows for a type-II configuration, does not lead to non-Abelian source-free solutions.

8. Higher subgroups

Out of the subgroups G_5 we selected only the one

$$\{P_1, P_2, P_0-P_3, L_2+K_1, L_1-K_2\}$$

which is the symmetry group of the plane waves of electrodynamics. The result is that not only non-Abelian source-free solutions of the Yang–Mills equations, but non-Abelian configurations in general are excluded for this symmetry. Thus we only have the embedding of the plane waves of electrodynamics, which is Abelian of course. Further details have been published elsewhere (Basler and Hädicke 1984a).

There are ten subgroups G_6 of the Poincaré group and all of them were included. All these G_6 act on the whole Minkowski space, so they are multiply transitive. The following five groups G_6 allow for Abelian configurations and source-free solutions:

$$\{P_1, P_2, P_3, P_0, L_2+K_1, L_1-K_2\}$$
$$\{P_1, P_2, P_3, P_0, L_3, K_3\}$$
$$\{P_1, P_2, P_0-P_3, L_2+K_1, L_1-K_2, K_3\}$$
$$\{P_1, P_2, P_0-P_3, L_2+K_1, L_1-K_k, L_3\pm\tfrac{1}{4}(P_0+P_3)\}$$
$$\{P_1, P_2, P_0-P_3, L_2+K_1, L_1-K_2, L_3\cos\vartheta - K_3\sin\vartheta\}.$$

On the other hand a non-Abelian configuration with a symmetry group G_6 does exist only for the one group

$$\{P_1, P_2, P_3, P_0, L_3, K_3\}.$$

The following configuration belongs to this group:

$$\begin{aligned} &A_t = 0 \qquad A_x = -2ST^1 - 2RT^2 \qquad A_y = 2RT^1 - 2ST^2 \qquad A_z = 0 \\ &S = \text{constant} \qquad R = \text{constant}. \end{aligned} \tag{8.1}$$

Because the configuration (8.1) produces an external current we find that non-Abelian source-free solutions with a symmetry group G_6 are generally excluded.

For groups G_r, $r > 6$ there exist neither Abelian nor non-Abelian configurations.

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